

Hamilton 体系层合板动力响应半解析法*

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【摘要】 对时间方向引进正则变换函数, 给出了层合板动力学的 Hamilton 正则方程及其半解析法。通过分离变量, 就可以在板平面内采用通常的有限元离散, 而在板厚方向采用状态空间法给出解析解, 且通过传递矩阵, 保证了层间位移和应力的连续, 建立了层合板上下表面相变量的关系式, 利用打靶法进行求解。数值算例表明: 本文方法未知量少, 精度高。

关键词: Hamilton 正则方程 半解析法 传递矩阵法 动力响应 层合板

1 前言

由于复合材料横向剪切模量较低, 剪切效应比较严重^[1~5], 因此人们提出许多考虑剪切效应的计算方法。本文从 Hellinger-Reissner 变分原理以及修正后导出的 Hamilton 正则方程出发, 在时间方向引进正则变换函数, 给出一种求解层合板动力响应的半解析法。该法在板平面内采用通常的有限元离散, 而在板厚方向采用状态空间法给出解析解答, 且通过传递矩阵, 保证了层间位移和应力的连续, 建立了层合板上下表面相变量的关系式, 利用打靶法进行求解。

2 层合板动力学的 Hamilton 正则方程

$$\frac{\partial}{\partial t} = \begin{bmatrix} 0 & G \\ F & 0 \end{bmatrix} \cdot D + f_1 - f_2 \quad (1)$$

其中 $D = [u \ v \ \alpha_z \ \tau_{xz} \ \tau_{yz} \ w]^T$, $f_1 = [0 \ 0 \ \rho_w^{\circ\circ} \ \rho_u^{\circ\circ} \ \rho_v^{\circ\circ} \ 0]^T$

$$f_2 = [0 \ 0 \ f_z \ f_x \ f_y \ 0]^T, \quad G = \begin{bmatrix} a_{10} & - & a_{11} & - & \partial/\partial\alpha \\ - & a_{11} & a_{12} & - & \partial/\partial\gamma \\ - & \partial/\partial\alpha & - & \partial/\partial\gamma & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} - & a_2 \frac{\partial^2}{\partial \alpha^2} - & a_{13} \frac{\partial^2}{\partial \gamma^2} - & 2a_{14} \frac{\partial^2}{\partial \alpha \partial \gamma} & - & a_{14} \frac{\partial^2}{\partial \alpha^2} - & a_{15} \frac{\partial^2}{\partial \gamma^2} - & (a_3 + a_{13}) \frac{\partial^2}{\partial \alpha \partial \gamma} & a_1 \frac{\partial}{\partial \alpha} + & a_{16} \frac{\partial}{\partial \gamma} \\ - & a_{14} \frac{\partial^2}{\partial \alpha^2} - & a_{15} \frac{\partial^2}{\partial \gamma^2} - & (a_3 + a_{13}) \frac{\partial^2}{\partial \alpha \partial \gamma} & - & a_{13} \frac{\partial^2}{\partial \alpha^2} - & a_4 \frac{\partial^2}{\partial \gamma^2} - & 2a_{15} \frac{\partial^2}{\partial \alpha \partial \gamma} & a_5 \frac{\partial}{\partial \gamma} + & a_{16} \frac{\partial}{\partial \alpha} \\ & a_1 (\partial/\partial\alpha) + & a_{16} (\partial/\partial\gamma) & & & a_5 (\partial/\partial\gamma) + & a_{16} (\partial/\partial\alpha) & & & a_7 \end{bmatrix}$$

$$a_1 = -c_{13}/c_{33}, \quad a_2 = c_{11} - c_{13}^2/c_{33}, \quad a_3 = c_{12} - c_{13}c_{23}/c_{33}, \quad a_4 = c_{22} - c_{23}^2/c_{33}$$

$$a_5 = -c_{13}/c_{33}, \quad a_6 = c_{66}, \quad a_7 = 1/c_{33}, \quad a_8 = 1/c_{55}, \quad a_9 = 1/c_{44}$$

$$a_{10} = c_{44}/(c_{44}c_{55} - c_{45}^2), \quad a_{11} = c_{45}/(c_{44}c_{55} - c_{45}^2), \quad a_{12} = c_{55}/(c_{44}c_{55} - c_{45}^2)$$

$$a_{13} = (c_{66} - c_{36}^2/c_{33}), \quad a_{14} = c_{16} - c_{13}c_{36}/c_{33}, \quad a_{15} = c_{26} - c_{23}c_{36}/c_{33}, \quad a_{16} = -c_{36}/c_{33}$$

式 (1) 是典型的 Hamilton 正则方程^[6], 但同时又含有时间的二次导数直接求解很困难。

为此提出正则变换法。不妨以 Wilson- θ 法为例, 取正则变换函数为

$$D_{t+\theta\Delta t} = 6(D_{t+\theta\Delta t} - D_t)/(\theta\Delta t)^2 - 6\dot{D}_t/(\theta\Delta t) - 2\ddot{D}_t \quad (2)$$

将式 (2) 代入式 (1) 就得到修正后的动力学的 Hamilton 正则方程为

$$\frac{\partial}{\partial} D_{t+\theta\Delta t} = \left[\begin{array}{c|c} 0 & G^* \\ \hline F^* & 0 \end{array} \right] \cdot D_{t+\theta\Delta t} + f_{t+\theta\Delta t} \quad (3)$$

其中

$$\begin{aligned} G^* &= G + G_t, \quad F^* = F + F_t \\ G_t &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6\rho/(\theta\Delta t)^2 \end{bmatrix}, \quad F_t = \begin{bmatrix} 6\rho/(\theta\Delta t)^2 & 0 & 0 \\ 0 & 6\rho/(\theta\Delta t)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ f_{t+\theta\Delta t} &= [0 \quad 0 \quad f_z^* \quad f_x^* \quad f_y^* \quad 0]^T \\ f_z^* &= f_z + 6\rho u_t/(\theta\Delta t)^2 + 6\rho w_t/(\theta\Delta t) + 2\rho \ddot{w}_t \\ f_x^* &= f_x + 6\rho u_t/(\theta\Delta t)^2 + 6\rho u_t/(\theta\Delta t) + 2\rho \ddot{u}_t \\ f_y^* &= f_y + 6\rho v_t/(\theta\Delta t)^2 + 6\rho v_t/(\theta\Delta t) + 2\rho \ddot{v}_t \end{aligned}$$

则:

$$\partial_{t+\theta\Delta t}/\partial = J^{-1} \partial H / \partial_{t+\theta\Delta t} \quad (4)$$

其中 $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, I 为单位阵, 而且有性质 $J^{-1} = J^T = J$, $JHJ = H^T$ 。

这说明混合状态是一种很基本的 Hamiltonian 结构, 它们的力学意义是它们构成辛几何空间, 当沿 z 方向介质模量发生变化时或两种介质接触时, 这 6 个相变量 ($u, v, \sigma_z, \tau_{xz}, \tau_{yz}, w$) 必须是连续的, 而 ($\sigma_x, \sigma_y, \tau_{xy}$) 则可以是间断的。

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} a_2(\partial/\partial x) + a_{14}(\partial/\partial y) & a_3(\partial/\partial y) + a_{14}(\partial/\partial x) & -a_1 \\ a_3(\partial/\partial x) + a_{15}(\partial/\partial y) & a_4(\partial/\partial y) + a_{15}(\partial/\partial x) & -a_5 \\ a_{14}(\partial/\partial x) + a_{13}(\partial/\partial y) & a_{15}(\partial/\partial y) + a_{13}(\partial/\partial x) & -a_{16} \end{bmatrix} \begin{bmatrix} u \\ v \\ \sigma_x \end{bmatrix} \quad (5)$$

3 混合状态元、Hamiltonian 元及其控制方程的解析解

对层合板任一沿 $x-y$ 平面离散, z 方向保留原形式, t 时刻相变量场设为

$$u = [N(x, y)]\{u(z)\}, \quad v = [N(x, y)]\{v(z)\}, \quad \sigma_z = [N(x, y)]\{\sigma(z)\}$$

$$\tau_{xz} = [N(x, y)]\{\tau_{xz}(z)\}, \quad \tau_{yz} = [N(x, y)]\{\tau_{yz}(z)\}, \quad w = [N(x, y)]\{w(z)\}$$

采用任意四边形单元形函数 $N_i(\xi, \eta) = (1/4)(1 + \xi\xi)(1 + \eta\eta)$, 取变分并分步积分有

$$C^e \partial_{t+\theta\Delta t}/\partial = K^e D_{t+\theta\Delta t} + P_{t+\theta\Delta t}^e \quad (6)$$

式中

$$\begin{aligned} C^e &= \iint \text{diag}(N^T N, N^T N, N^T N, N^T N, N^T N, N^T N) dx dy, \quad K^e = \left[\begin{array}{c|c} 0 & G^{*e} \\ \hline F^{*e} & 0 \end{array} \right] \\ G^{*e} &= \iint \begin{bmatrix} a_{10} N^T N & -a_{11} N^T N & -N^T (\partial/\partial x) \\ -a_{11} N^T N & a_{12} N^T N & -N^T (\partial/\partial y) \\ (\partial^T/\partial x) N & (\partial^T/\partial y) N & 6\rho N^T N / (\theta\Delta t)^2 \end{bmatrix}, \quad F^{*e} = \iint \begin{bmatrix} A' & B' & C' \\ D' & E' & F' \\ G' & H' & K' \end{bmatrix} dx dy \end{aligned}$$

式中

$$\begin{aligned} A' &= a_2 \frac{\partial^T}{\partial x} \frac{\partial}{\partial x} + a_{13} \frac{\partial^T}{\partial y} \frac{\partial}{\partial y} + a_{14} \frac{\partial^T}{\partial x} \frac{\partial}{\partial y} + a_{14} \frac{\partial^T}{\partial y} \frac{\partial}{\partial x} + \frac{6\rho}{\theta\Delta t^2} N^T N \\ B' &= a_{14} \frac{\partial^T}{\partial x} \frac{\partial}{\partial x} + a_{15} \frac{\partial^T}{\partial y} \frac{\partial}{\partial y} + a_{13} \frac{\partial^T}{\partial y} \frac{\partial}{\partial x} + a_3 \frac{\partial^T}{\partial x} \frac{\partial}{\partial y} \end{aligned}$$

$$\begin{aligned}
C' &= -[a_{11}(\partial^T/\partial x)N + a_{16}(\partial^T/\partial y)N] \\
D' &= a_{14}\frac{\partial^T}{\partial x}\frac{\partial^T}{\partial x} + a_{15}\frac{\partial^T}{\partial y}\frac{\partial^T}{\partial y} + a_{13}\frac{\partial^T}{\partial x}\frac{\partial^T}{\partial y} + a_{33}\frac{\partial^T}{\partial y}\frac{\partial^T}{\partial x} \\
E' &= a_{13}\frac{\partial^T}{\partial x}\frac{\partial^T}{\partial x} + a_{44}\frac{\partial^T}{\partial y}\frac{\partial^T}{\partial y} + a_{15}\frac{\partial^T}{\partial x}\frac{\partial^T}{\partial y} + a_{15}\frac{\partial^T}{\partial y}\frac{\partial^T}{\partial x} + \frac{6\rho}{\theta^2\Delta t^2}N^TN \\
F' &= -[a_{55}(\partial^T/\partial y)N + a_{16}(\partial^T/\partial x)N], \quad G' = a_{11}N^T(\partial/\partial x) + a_{16}N^T(\partial/\partial y) \\
H' &= a_{55}N^T(\partial/\partial y) + a_{16}N^T(\partial/\partial x), \quad K' = a_{77}N^TN, \quad P^e = P_1^e + P_2^e \\
P_1^e &= \begin{bmatrix} 0 & 0 & -\iint T f_z^* dx dy & -\iint T f_x^* dx dy & -\iint T f_y^* dx dy & 0 \end{bmatrix}^T \\
P_2^e &= \begin{bmatrix} 0 & 0 & -\int_{s_0} T \bar{p}_z ds & -\int_{s_0} T \bar{p}_x ds & -\int_{s_0} T \bar{p}_y ds & 0 \end{bmatrix}^T
\end{aligned}$$

通过上面沿 $x-y$ 平面的离散, 就得到层合板中第 m 层混合状态 Hamiltonian 元控制方程

$$C_m \mathcal{D}_m^{t+\theta\Delta t}/\partial = K_m D_m^{t+\theta\Delta t} + P_m^{t+\theta\Delta t} \quad (7)$$

其中 $C_m = \sum_e C_m^e$, $K_m = \sum_e K_m^e$, $P_m = \sum_e P_m^e$. 式 (7) 两端同乘以逆阵 C_m^{-1} 有

$$\mathcal{D}_m^{t+\theta\Delta t}/\partial = A_m D_m^{t+\theta\Delta t} + B_m^{t+\theta\Delta t} \quad (8)$$

其解析解为 $\mathcal{D}_m^{t+\theta\Delta t}(z) = T_m(z)D_m^{t+\theta\Delta t}(0) + G_m^{t+\theta\Delta t}$ (9)

其中 $T_m(z) = e^{A_m z}$, 而 $G_m^{t+\theta\Delta t}(z)$ 的计算可变换 $\xi = 2\tau/z - 1$, $\tau = (1+\xi)z/2$, 用高斯积分求积

$$G_m^{t+\theta\Delta t} = \int_0^z A_m(z-\tau)B_m^{t+\theta\Delta t}(\tau)d\tau = \int_{-1}^{+1} e^{A_m \frac{z(1-\xi)/2}{} } B_m^{t+\theta\Delta t} \left[\frac{z(1+\xi)}{2} \right] \frac{z}{2} d\xi \quad (10)$$

实际上, 式 (9) 对任一层都成立, 取两典型层 $m=1, p$ 分别有

$$\begin{cases} D_1^{t+\theta\Delta t}(h_1) = T_1(h_1)D_1^{t+\theta\Delta t}(0) + G_1^{t+\theta\Delta t}(h_1) \\ D_p^{t+\theta\Delta t}(h_p) = T_p(h_p)D_p^{t+\theta\Delta t}(0) + G_p^{t+\theta\Delta t}(h_p) \end{cases} \quad (11)$$

考虑层间位移及应力连续条件 $D_j^{t+\theta\Delta t}(h_j) = D_{j+1}^{t+\theta\Delta t}(0)$, 利用传递矩阵法层层传递, 最后得

$$\begin{Bmatrix} \{q(h)\} \\ \{P(h)\} \end{Bmatrix}_{t+\theta\Delta t} = \begin{bmatrix} T_{11}^* & T_{12}^* \\ T_{21}^* & T_{22}^* \end{bmatrix} \begin{Bmatrix} \{q(0)\} \\ \{P(0)\} \end{Bmatrix}_{t+\theta\Delta t} + \begin{Bmatrix} \{G\}_q \\ \{G\}_p \end{Bmatrix}_{t+\theta\Delta t} \quad (12)$$

其中 $\{q(0)\}$ 、 $\{P(0)\}$ 、 $\{q(h)\}$ 、 $\{P(h)\}$ 分别为层合板上下表面的位移和应力向量, $\{G\}_q$ 、 $\{G\}_p$ 是与边界条件及外载相关的等效荷载. 代入上下表面的已知应力边界条件, 最后得

$$[H]\{q(0)\}_{t+\theta\Delta t} = \{Q\}_{t+\theta\Delta t} \quad (13)$$

其中 $[H] = [T_{21}^*]$, $\{Q\}_{t+\theta\Delta t} = \{P(h)\}_{t+\theta\Delta t} - T_{22}^* \{P(0)\}_{t+\theta\Delta t} - \{G\}_{p,t+\theta\Delta t}$

4 数值算例和讨论

为了考察本方法的精度, 考虑文献 [2] 中层合板, 其弹性常数及几何尺寸为: $E_{11} = 25E_{22}$, $E_{22} = E_{33} = 2.1 \times 10^6 \text{ N/cm}^2$, $G_{12} = G_{13} = G_{23} = 0.5E_{22}$, $\mu_{12} = \mu_{13} = \mu_{23} = 0.25$, $\rho = 8 \times 10^{-6} \text{ N} \cdot \text{s}^2/\text{cm}^4$, $a/h = 5$, $h = 5 \text{ cm}$, 受正弦分布荷载 $q = q_0 \sin(\pi x/a) \sin(\pi y/a)$ 作用. 取时间步长 $\Delta t = 5 \mu\text{s}$, 无量纲挠度和应力为 $\bar{w} = 1000E_{22}W/(q_0 s^2)$, $\bar{\alpha} = 10\alpha/(q_0 s^2)$, $s = a/h_0$.

从图 1 和图 2 可以看出: 本法所得的位移和应力与高阶理论 [2] 吻合得很好.

考虑 (019010) 三层层合板 (Pagano [5]), 其弹性常数及几何尺寸为: $E_{11} = 25E_{22}$, $E_{22} = E_{33} = 2.1 \times 10^6 \text{ N/cm}^2$, $G_{12} = G_{13} = 0.5E_{22}$, $G_{23} = 0.2E_{22}$, $\mu_{12} = \mu_{13} = \mu_{23} = 0.25$, $\rho = 8 \times 10^{-6} \text{ N} \cdot \text{s}^2/\text{cm}^4$, $a/h = 4$, $h = 5 \text{ cm}$. 网格划分 4×4 , 时间步长 $\Delta t = 2 \mu\text{s}$. 无量纲应力: $\bar{\alpha} = 10\alpha/(q_0 s^2)$, $(\bar{\tau}_{xz}, \bar{\tau}_{yz}) = (\tau_{xz}, \tau_{yz}) \times 10/(q_0 s)$, $s = a/h_0$.

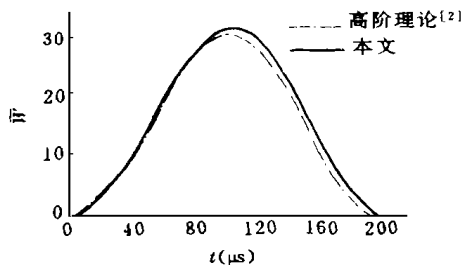
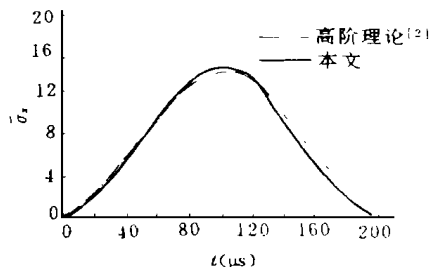
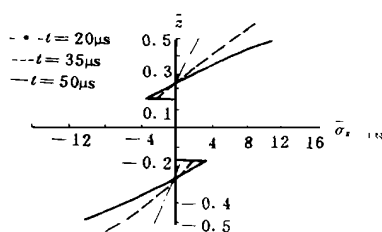
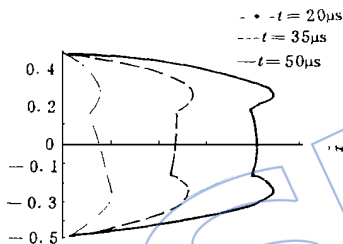
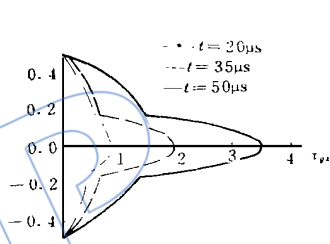
图1 $\bar{W}(a/2, a/2, 0, t)$ 变化曲线图2 $\bar{O}_x(a/2, a/2, 0, t)$ 变化曲线图3 $\bar{O}_x(a/2, a/2)$ 沿 z 向变化图4 $\bar{O}_{xz}(0, a/2)$ 沿 z 向变化图5 $\bar{O}_{yz}(a/2, 0)$ 沿 z 向变化

图3~5 分别给出 $\bar{O}_x$, $\bar{O}_{xz}$, $\bar{O}_{yz}$ 在不同时刻沿板厚方向变化曲线。

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